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MINIMAX PROCEDURES FOR TWO-VALUED DECISION PROBLEMS WHEN THE SIZE OF SAMPLE IS FIXED

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By

S. G. ALLEN, JR.

TECHNICAL REPORT NO. 3
APRIL 20, 1951

JUN 28 1951

PREPARED UNDER CONTRACT N6onr-25126
(NR-042-002)
ALBERT H. BOWKER, Director
FOR
OFFICE OF NAVAL RESEARCH

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MINIMAX PROCEDURES FOR TWO-VALUED DECISION PROBLEMS WHEN THE SIZE OF SAMPLE IS FIXED*

by

S. G. Allen, Jr.

1. The Minimax Solution for the Case of a Simple Dichotomy. The problem to be considered is a statistical decision procedure for choosing between two alternative actions, A_1 and A_2, after taking n independent observations on a random variable x. The probability density $p(x,\theta)$ of x is known except for the value of a real parameter θ which is assumed to be one-dimensional.1/ The parameter space Ω is partitioned into two subspaces Ω_1 and Ω_2 such that decision A_1 is preferred if $\theta \varepsilon \Omega_1$ and A_2 preferred if $\theta \varepsilon \Omega_2$. The costs of decisions A_1 and A_2 are, respectively,

$$1.1 \quad w_1(\theta) \begin{cases} = 0, & \theta \varepsilon \Omega_1 \\ \geqslant 0, & \theta \varepsilon \Omega_2 \end{cases}$$

and

$$1.2 \quad w_2(\theta) \begin{cases} \geqslant 0, & \theta \varepsilon \Omega_1 \\ = 0, & \theta \varepsilon \Omega_2 . \end{cases}$$

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* The motivating idea for Theorem A of this report is to be found in a lot acceptance sampling procedure proposed in an unpublished paper by Mr. Norman Rudy of Sacramento State College. Discussion with members of the Department of Statistics, Stanford University, in particular with Professor M. A. Girshick, was most beneficial in the formulation of the present draft of the report.

1/ The main results of this paper will be based on the assumption that the probability distribution of x is continuous. The necessary modification of the results for the case of discrete probability distributions will be discussed in a concluding section of the paper.

If the parameter space consists of a simple dichotomy (i.e. Ω_1 consists of the single point θ_1, and Ω_2, the single point θ_2), the minimax procedure is well known. The relevant statistic is the likelihood ratio

$$1.3 \quad \lambda = \lambda(\theta_1, \theta_2) = \frac{\prod_{i=1}^{n} p(x_i, \theta_2)}{\prod_{i=1}^{n} p(x_i, \theta_1)} \text{ ,}$$

and the decision procedure which minimizes the maximum expected cost is to

1.4 choose A_1, if $\lambda \leqq c$, and

1.5 choose A_2, if $\lambda > c$,

where the criterion c satisfies the relation

$$1.6 \quad w_1(\theta_2) \Pr(\lambda \leqq c|\theta_2) = w_2(\theta_1) \Pr(\lambda > c|\theta_1) \text{ .}$$

This value of c is

$$1.7 \qquad c = \frac{w_2(\theta_1)g}{w_1(\theta_2)(1-g)} \text{ ,}$$

where g is the least favorable a priori probability that $\theta = \theta_1$.

2. The Minimax Solution to a More General Problem. The assumption that Ω_1 and Ω_2 each consists of a single point is often a very artificial one. A more general formulation of the two-valued decision problem is the following: Ω is an interval, $\theta \varepsilon \Omega_1$ if $\theta \leqq \theta_0$, and $\theta \varepsilon \Omega_2$ if $\theta \geqq \theta_0$, where the costs of decisions A_1 and A_2

are given by 1.1, 1.2, and

$$2.1 \qquad w_1(\theta_o) = w_2(\theta_o) = 0. \; ^{2/}$$

This statement of the problem avoids the often unrealistic postulate of an "indifference interval", i.e. an interval such that decisions A_1 and A_2 are either costless or equally costly if θ assumes a value in this interval. However, in this more general formulation, is the likelihood ratio test for some $\theta_1 \varepsilon \Omega_1$ and $\theta_2 \varepsilon \Omega_2$ still a minimax decision procedure?

The following theorem supplies the answer.

Theorem A. Let

$$2.2 \quad R_1(c,\theta_1,\theta_2,\theta) = w_1(\theta) \Pr[\lambda(\theta_1,\theta_2) \leqq c|\theta] \text{ and}$$

$$2.3 \quad R_2(c,\theta_1,\theta_2,\theta) = w_2(\theta) \Pr[\lambda(\theta_1,\theta_2) > c|\theta], \text{ where}$$

$\theta_1 \varepsilon \Omega_1$ and $\theta_2 \varepsilon \Omega_2$. Then the likelihood ratio test is a minimax procedure if and only if there exists a triple $(c_o,\theta_1^o,\theta_2^o)$ such that

$$2.4 \quad \max_{\theta \varepsilon \Omega_2} R_1(c_o,\theta_1^o,\theta_2^o,\theta) = R_1(c_o,\theta_1^o,\theta_2^o,\theta_2^o) = R_2(c_o,\theta_1^o,\theta_2^o,\theta_1^o)$$
$$= \max_{\theta \varepsilon \Omega_1} R_2(c_o,\theta_1^o,\theta_2^o,\theta) .$$

Proof of necessity:

Suppose the likelihood ratio test for some $\theta_1^o \varepsilon \Omega_1$, $\theta_2^o \varepsilon \Omega_2$, and $c = c^o$ is minimax. Then 2.4 follows.

2/ It seems only reasonable to assume that $\inf_{\theta \varepsilon \Omega_2} w_1(\theta) = w_1(\theta_o)$ and $\inf_{\theta \varepsilon \Omega_1} w_2(\theta) = w_2(\theta_o)$. With this in view, there is no loss in generality in requiring 2.1.

Proof of sufficiency:

Let F_o denote the likelihood ratio test procedure associated with the triple $(c_o, \theta_1^o, \theta_2^o)$ satisfying 2.4. Then the expected cost of this procedure is given by

$$2.5 \quad R(F_o, G) = \int_{\Omega_2} w_1(\theta) \; Pr[\lambda(\theta_1^o, \theta_2^o) \leq c_o | \theta] \; dG(\theta)$$

$$+ \int_{\Omega_1} w_2(\theta) \; Pr[\lambda(\theta_1^o, \theta_2^o) > c_o | \theta] \; dG(\theta)$$

where $dG(\theta)$ denotes any a priori probability measure over Ω. Since $R_1(c_o, \theta_1^o, \theta_2^o, \theta_2^o)$ and $R_2(c_o, \theta_1^o, \theta_2^o, \theta_1^o)$ are the maxima of the integrands appearing in 2.5, it follows that

$$2.6 \quad R(F_o, G) \leq \int_{\Omega_2} R_1(c_o, \theta_1^o, \theta_2^o, \theta_2^o) \; dG(\theta) + \int_{\Omega_1} R_2(c_o, \theta_1^o, \theta_2^o, \theta_1^o) \; dG(\theta)$$

$$= R_1(c_o, \theta_1^o, \theta_2^o, \theta_2^o) = R_2(c_o, \theta_1^o, \theta_2^o, \theta_1^o) .$$

If G is a distribution concentrating all probability at θ_1^o and θ_2^o, 2.6 is a strict equality. Therefore

$$2.7 \quad \max_G R(F_o, G) = R_1(c_o, \theta_1^o, \theta_2^o, \theta_2^o) = R_2(c_o, \theta_1^o, \theta_2^o, \theta_1^o).$$

Let G_o be the a priori distribution given by

$$2.8 \quad g = Pr(\theta = \theta_1^o)$$

$$2.9 \quad 1-g = Pr(\theta = \theta_2^o)$$

where g satisfies

$$2.10 \qquad c_o = \frac{w_2(\theta_1^o)g}{w_1(\theta_2^o)(1-g)} \quad .$$

Clearly F_o is the Bayes procedure against G_o. Therefore

$$2.11 \qquad \min_F R(F,G_o) = R(F_o,G_o) \ ,$$

where F is any decision procedure for choosing between A_1 and A_2. Since G_o is a distribution for which 2.6 is a strict equality, the minimax property of F_o follows.

3. Assumptions Under Which the Likelihood Ratio Test Is a Minimax Procedure. It is now of interest to examine a class of distribution functions and a class of cost functions for which condition 2.4 of Theorem A is satisfied.

Consider a family of probability measures defined by

$$3.1 \quad dL_o(x|\theta) = \frac{1}{\omega_o(\theta)} \ e^{\theta x} \, d\Psi_o(x) \ ,$$

where $\Psi_o(x)$ is a measure defined over an interval X, and where

$$3.2 \qquad \omega_o(\theta) = \int_{-\infty}^{\infty} e^{\theta x} \, d\Psi_o(x)$$

is a bounded function of θ defined over an interval Ω.[3] If u is the sum of n independent random variables each distributed according to 3.1, then u is distributed according to

[3] The properties of this family of distribution functions are given an extensive discussion in Section 3 of Girshick and Savage, "Bayes and Minimax Estimates for Quadratic Loss Functions", Technical Report No. 5, Contract No. N6-ONR-251 Task III, Department of Statistics, Stanford University, November 21, 1950.

$$3.3 \quad dL(u|\theta) = \frac{1}{\omega(\theta)} e^{\theta u} d\Psi(u) .$$

This is again the distribution family introduced in 3.1 and 3.2, with $\Psi(u)$ defined over an interval U.

In view of the form of the distributions 3.1 and 3.3, it is simple to show that for every number c such that $\lambda \lessgtr c$ (where λ is defined in 1.3) there exists a unique number k such that $u \lessgtr k$. This correspondence permits the functions 2.2 and 2.3 to be rewritten as

$$3.4 \qquad R_1(k,\theta) = w_1(\theta)\ L(k|\theta)$$

$$3.5 \qquad R_2(k,\theta) = w_2(\theta)\ [1-L(k|\theta)] .$$

These functions have the following important property:

<u>Lemma 1.</u>

If $\max\limits_{\theta\epsilon\Omega_2} R_1(k,\theta)$ and $\max\limits_{\theta\epsilon\Omega_1} R_2(k,\theta)$ exist, then they are, respectively, monotonic increasing and decreasing functions of k.

Proof:

Let $\max\limits_{\theta\epsilon\Omega_2} R_1(k,\theta) = w_1(\theta')L(k|\theta'),\ \theta'\epsilon\Omega_2$

and $\max\limits_{\theta\epsilon\Omega_1} R_2(k,\theta) = w_2(\theta'')[1-L(k|\theta'')],\ \theta''\epsilon\Omega_1$.

Obviously $L(k|\theta)$ is an increasing function of k for any θ.

Thus for Δ positive

$$w_1(\theta')\ L(k|\theta') \leqq w_1(\theta')\ L(k+\Delta|\theta') \leqq \max_{\theta\epsilon\Omega_2} w_1(\theta)\ L(k+\Delta|\theta) \equiv \max_{\theta\epsilon\Omega_2} R_1(k+\Delta,\theta)$$

and

$$w_2(\theta'')[1-L(k|\theta'')] \leqq w_2(\theta'')[1-L(k-\Delta|\theta'')] \leqq \max_{\theta\epsilon\Omega_1} w_2(\theta)[1-L(k-\Delta|\theta'')]$$

$$\equiv \max_{\theta\epsilon\Omega_1} R_2(k-\Delta,\theta),$$

as desired.

It is now possible to prove the following

<u>Theorem B</u>. Let $\Psi(u)$, $w_1(\theta)$, and $w_2(\theta)$ be continuous throughout any finite intervals in $U = (a,b)$, Ω_2, and Ω_1, respectively.[4] Let

$$3.6 \qquad \lim_{k\to a} R_1(k,\theta) = \lim_{k\to b} R_2(k,\theta) = 0$$

uniformly in θ. Then if the $\max_{\theta\epsilon\Omega_2} R_1(k,\theta)$ and $\max_{\theta\epsilon\Omega_1} R_2(k,\theta)$ exist for some $k \epsilon U$, there exists a unique value of $k \epsilon U$ such that

$$3.7 \qquad \max_{\theta\epsilon\Omega_2} R_1(k,\theta) = \max_{\theta\epsilon\Omega_1} R_2(k,\theta) .$$

Proof:

Suppose for some $k = k_o \epsilon U$, $\max_{\theta\epsilon\Omega_2} R_1(k_o,\theta) > \max_{\theta\epsilon\Omega_1} R_2(k_o,\theta)$. For $k < k_o$, $R_1(k,\theta)$ is uniformly bounded by $\max_{\theta\epsilon\Omega_2} R_1(k_o,\theta)$. Since $R_1(k,\theta)$ is continuous in k and θ, the $\max_{\theta\epsilon\Omega_2} R_1(k,\theta)$ exists and is continuous for $k < k_o$. Certainly the $\sup_{\theta\epsilon\Omega_1} R_2(k,\theta)$ is continuous in k, and for $k < k_o$ $\sup_{\theta\epsilon\Omega_1} R_2(k,\theta) \geqslant \max_{\theta\epsilon\Omega_1} R_2(k_o,\theta)$. Since by 3.6 $\max_{\theta\epsilon\Omega_2} R_2(k,\theta) \to 0$ as $k \to a$, there must be some $k \epsilon (a,k_o)$ such that $\max_{\theta\epsilon\Omega_2} R_1(k,\theta) = \sup_{\theta\epsilon\Omega_1} R_2(k,\theta)$, or, actually, $\max_{\theta\epsilon\Omega_2} R_1(k,\theta) = \max_{\theta\epsilon\Omega_1} R_2(k,\theta)$. Furthermore in view of Lemma 1 there is only one such value of k. If on the other hand $\max_{\theta\epsilon\Omega_1} R_2(k,\theta) > \max_{\theta\epsilon\Omega_2} R_1(k,\theta)$ for some $k_o \epsilon U$, then, in like manner, the restriction 3.6, Lemma 1, and the continuity hypothesis imply the existence of a unique $k \epsilon (k_o,b)$ such that $\max_{\theta\epsilon\Omega_2} R_1(k,\theta) = \max_{\theta\epsilon\Omega_1} R_2(k,\theta)$.

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[4] The interval U may of course be the open interval $(-\infty, \infty)$.

Corollary 1. Let $\Psi(u)$, $w_1(\theta)$, and $w_2(\theta)$ be continuous and bounded over $U = (a,b)$, Ω_2, and Ω_1, respectively.[5] Then there exists a unique $k \varepsilon U$ such that $\max_{\theta\varepsilon\Omega_2} R_1(k,\theta) = \max_{\theta\varepsilon\Omega_1} R_2(k,\theta)$.

Corollary 2. Let

3.8 $\quad w_1(\theta) = O(e^{\delta_2\theta}) \qquad \delta_2 > 0$

3.9 $\quad w_2(\theta) = O(e^{-\delta_1\theta}) \qquad \delta_1 > 0$

be continuous functions throughout any finite intervals in Ω_2 and Ω_1, respectively. Let $\Psi(u)$ be continuous, positive, and less than $\Psi(\infty)$ for any real u. Then there exists a unique $k \varepsilon U$ such that $\max_{\theta\varepsilon\Omega_2} R_1(k,\theta) = \max_{\theta\varepsilon\Omega_1} R_2(k,\theta)$.

Proof:

For $\Delta > 0$,

$$(1)\quad \lim_{\theta\to\infty} w_1(\theta)L(k|\theta) \leqq \lim_{\theta\to\infty} e^{\delta_2\theta} \frac{\int_{-\infty}^{k} e^{\theta u}\, d\Psi(u)}{\int_{-\infty}^{\infty} e^{\theta u}\, d\Psi(u)} \cdot \text{constant}$$

$$\leqq \lim_{\theta\to\infty} \frac{\int_{-\infty}^{k} e^{(u+\delta_2)\theta}\, d\Psi(u)}{\int_{k+\delta_2+\Delta}^{\infty} e^{\theta u}\, d\Psi(u)} \cdot \text{constant}$$

$$\leqq \lim_{\theta\to\infty} \frac{e^{(\delta_2+k)\theta}}{e^{(\delta_2+\Delta+k)\theta}} \frac{\int_{-\infty}^{k} d\Psi(u)}{\int_{k+\delta_2+\Delta}^{\infty} d\Psi(n)} \cdot \text{constant} = 0, \text{ and}$$

---

[5] $\Psi(u)$ is of course always bounded.

(2) $$\lim_{\theta\to-\infty} w_2(\theta)[1-L(k|\theta)] \leqq \lim_{\theta\to-\infty} e^{-\delta_1\theta} \frac{\int_k^\infty e^{\theta u}\, d\Psi(u)}{\int_{-\infty}^{\infty} e^{\theta u}\, d\Psi(u)} \cdot \text{constant}$$

$$\leqq \lim_{\theta\to-\infty} \frac{\int_k^\infty e^{(u-\delta_1)\theta}\, d\Psi(u)}{\int_{-\infty}^{k-\delta_1-\Delta} e^{u\theta}\, d\Psi(u)} \cdot \text{constant}$$

$$\leqq \lim_{\theta\to-\infty} \frac{e^{(k-\delta_1)\theta}\int_k^\infty d\Psi(u)}{e^{(k-\delta_1-\Delta)\theta}\int_{-\infty}^{k-\delta_1-\Delta} d\Psi(u)} \cdot \text{constant}$$

$$= 0 .$$

Since

(3) $$\lim_{\theta\to\infty} w_1(\theta)L(k|\theta) = \lim_{\theta\to-\infty} w_2(\theta)[1-L(k|\theta)] = w_1(\theta_o)L(k|\theta_o)$$

$$= w_2(\theta_o)[1-L(k|\theta_o)] = 0$$

for any k, the $\max_{\theta\in\Omega_2} R_1(k,\theta)$ and $\max_{\theta\in\Omega_1} R_2(k,\theta)$ exist for any k.

The $\min_{\theta\in\Omega_2} \omega(\theta)$ and $\min_{\theta\in\Omega_1} \omega(\theta)$ are positive constants. Therefore,

for $\theta\in\Omega_2$,

$$(4) \quad w_1(\theta)L(k|\theta) \leqq e^{\delta_2\theta} \int_{-\infty}^{k} e^{\theta u}\, d\Psi(u) \cdot \text{constant}$$

$$\leqq \int_{-\infty}^{k} e^{(\delta_2+u)(\theta-\theta_0+1)}\, e^{(\delta_2+u)(\theta_0-1)}\, d\Psi(u) \cdot \text{constant}$$

$$\leqq e^{(\delta_2+k)(\theta-\theta_0+1)} \int_{-\infty}^{k} e^{(\delta_2+u)(\theta_0-1)}\, d\Psi(u) \cdot \text{constant}$$

$$\leqq e^{\delta_2+k} \int_{-\infty}^{k} e^{(\delta_2+u)(\theta_0-1)}\, d\Psi(u) \cdot \text{constant},$$

and similarly, for $\theta \varepsilon \Omega_1$,

$$(5) \quad w_2(\theta)[1-L(k|\theta)] \leqq e^{k-\delta_1} \int_{k}^{\infty} e^{(u-\delta_1)(\theta_0+1)}\, d\Psi(u) \cdot \text{constant}.$$

The inequalities (4) and (5) demonstrate that 3.6 of Theorem B is satisfied. The conclusion of Corollary 2 follows.

Theorem B serves primarily as a criterion for determining whether or not minimax likelihood ratio tests exist in particular cases. The corollaries to Theorem B are interesting as applications of this criterion in the analysis of certain classes of sampling distributions and cost functions. Individual cases could of course be subjected to a much more direct analysis.

4. Remarks on the Discrete Case. The generality of the family of distributions introduced in 3.1 should not be overlooked: it includes many of the most important distributions encountered in statistics, such as the normal, binomial, and Poisson distributions. Suppose the distribution under consideration in this class

is a discrete one, and suppose that $\Psi(u)$ assumes jumps at each value of a denumerable sequence in which the values are placed in order of increasing magnitude. For example, in the binomial distribution, $u = 0,1,2,\ldots,n$. In general it will not be possible to find a value of k in such a sequence so that $\max_{\theta\epsilon\Omega_2} R_1(k,\theta) = \max_{\theta\epsilon\Omega_1} R_2(k,\theta)$. However, provided that all other conditions of Theorem B are satisfied except for the continuity of $\Psi(u)$, it will be possible to find a pair (k_o, k_o') such that

4.1 $$\max_{\theta\epsilon\Omega_2} R_1(k_o\theta) < \max_{\theta\epsilon\Omega_1} R_2(k_o,\theta)$$

4.2 $$\max_{\theta\epsilon\Omega_2} R_1(k_o',\theta) > \max_{\theta\epsilon\Omega_1} R_2(k_o',\theta) ,$$

where k_o' is the next larger value in the sequence than k_o. The minimax procedure is then a convex linear mixture of two procedures involving k_o and k_o', a mixture determined by

4.3 $$\max_{\theta\epsilon\Omega_2} [fR_1(k_o,\theta) + (1-f)R_1(k_o',\theta] = \max_{\theta\epsilon\Omega_1} [fR_2(k_o,\theta) + (1-f)R_2(k_o',\theta)],$$

where $0 < f < 1$.

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